This time: Fuzzy Logic and Fuzzy Inference

- Why use fuzzy logic?
- Tipping example
- Fuzzy set theory
- Fuzzy inference

What is fuzzy logic?

- A super set of Boolean logic
- Builds upon fuzzy set theory
- Graded truth. Truth values between True and False. Not everything is either/or, true/false, black/white, on/off etc.
- Grades of membership. Class of tall men, class of far cities, class of expensive things, etc.
- Lotfi Zadeh, UC/Berkely 1965. Introduced **FL to model uncertainty in natural language**. *Tall, far, nice, large, hot, ...*
- Reasoning using linguistic terms. Natural to express expert knowledge.

  If the weather is cold then wear warm clothing

Why use fuzzy logic?

Pros:

- Conceptually easy to understand w/ “natural” maths
- Tolerant of imprecise data
- Universal approximation: can model arbitrary nonlinear functions
- Intuitive
- Based on linguistic terms
- Convenient way to express expert and common sense knowledge

Cons:

- Not a cure-all
- Crisp/precise models can be more efficient and even convenient
- Other approaches might be formally verified to work

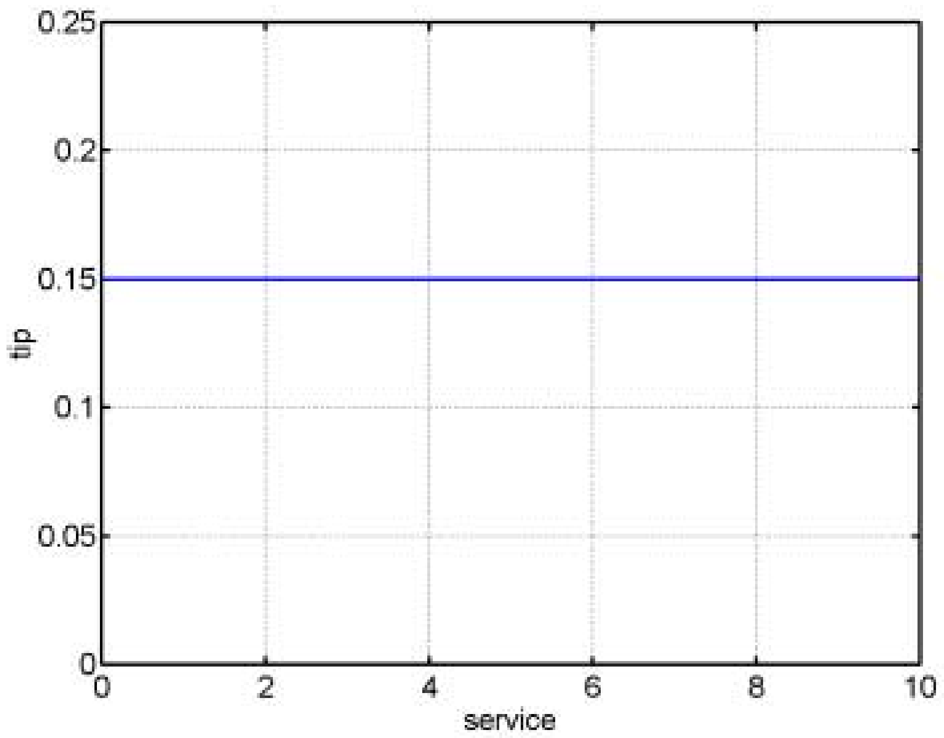
Tipping example

- **The Basic Tipping Problem:** Given a number between 0 and 10 that represents the quality of service at a restaurant what should the tip be?

  Cultural footnote: An average tip for a meal in the U.S. is 15%, which may vary depending on the quality of the service provided.

Tipping example: The non-fuzzy approach

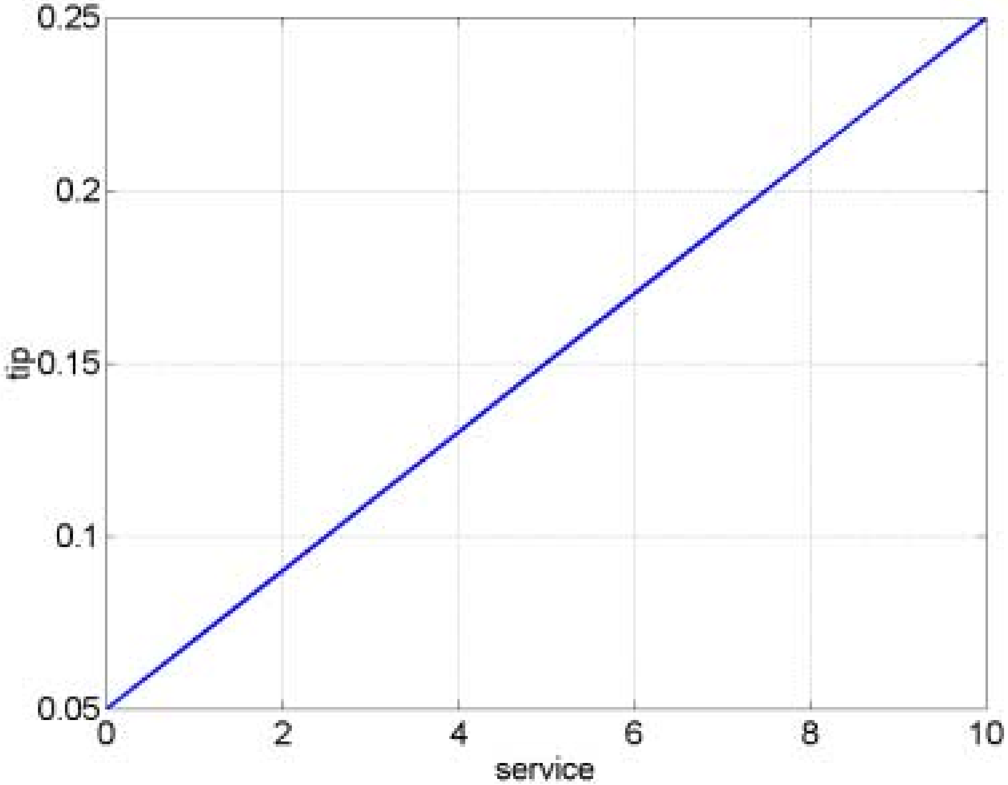
- Tip = 15% of total bill



- What about quality of service?

Tipping example: The non-fuzzy approach

- Tip = linearly proportional to service from 5% to 25%
  tip = 0.20/10*service+0.05



- What about quality of the food?

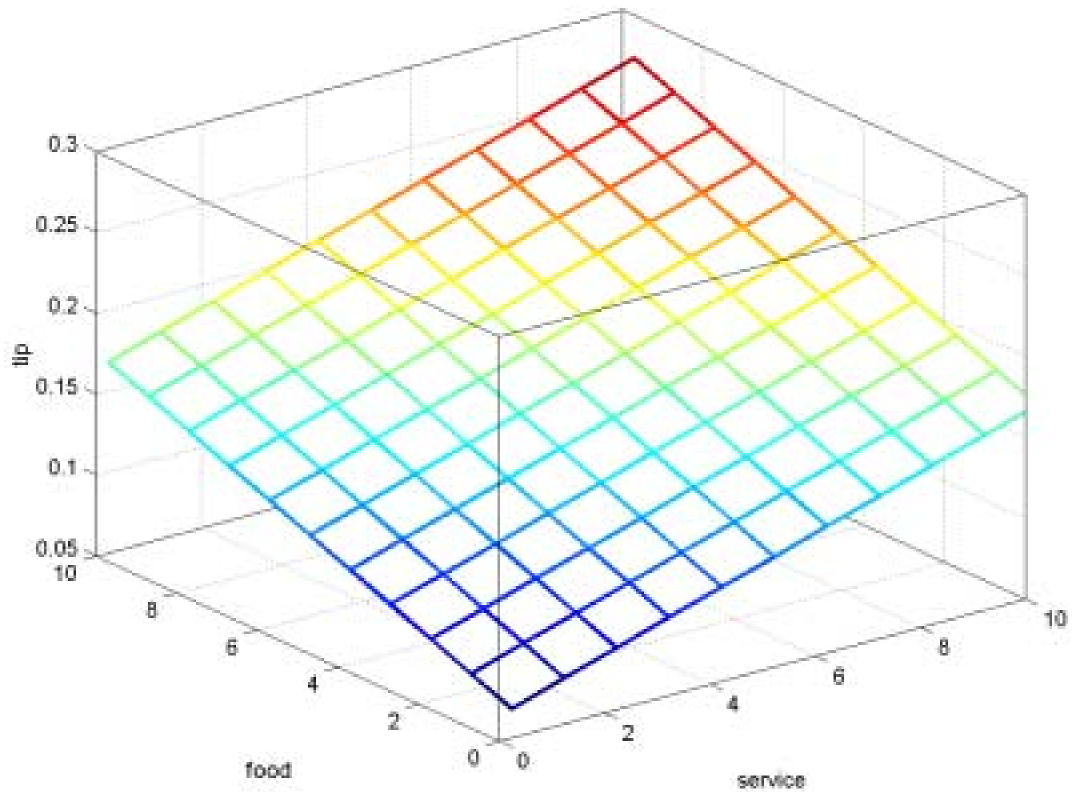
Tipping example: Extended

- **The Extended Tipping Problem:** Given a number between 0 and 10 that represents the quality of service <u>and the quality of the food</u>, at a restaurant, what should the tip be?

  How will this affect our tipping formula?

Tipping example: The non-fuzzy approach

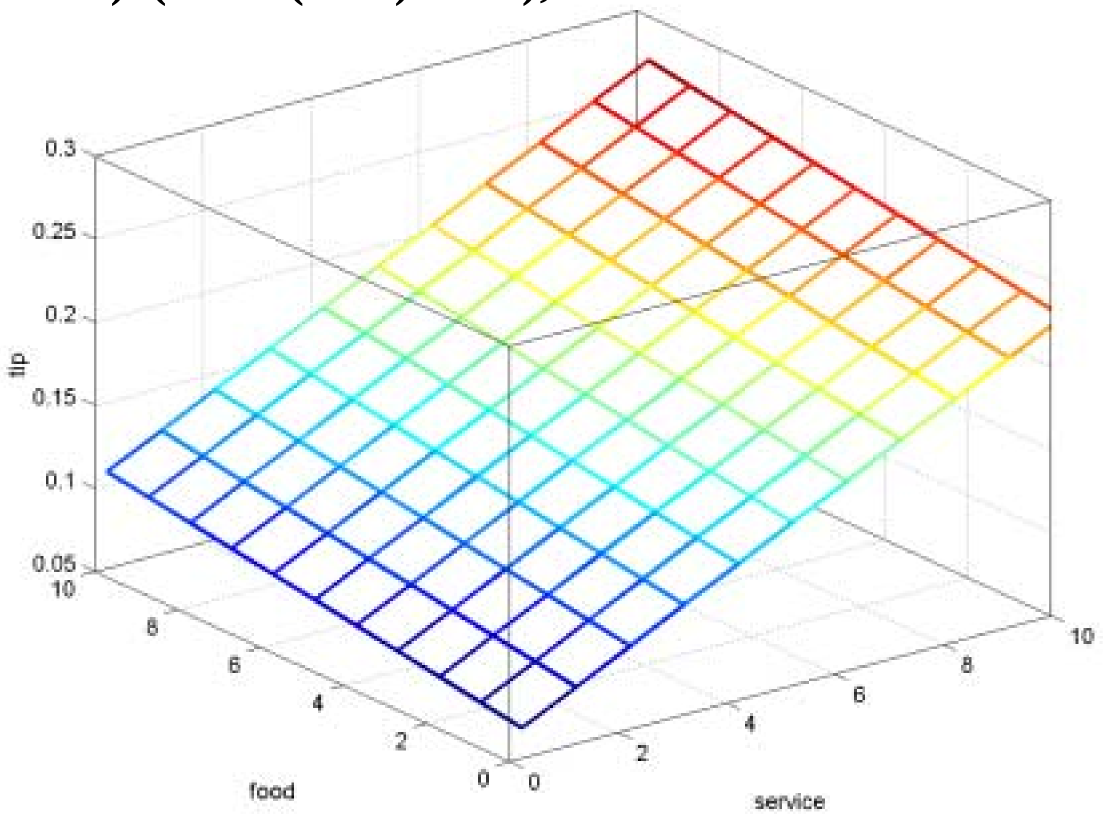
- Tip = 0.20/20*(service+food)+0.05



- We want service to be more important than food quality. E.g., 80% for service and 20% for food.

Tipping example: The non-fuzzy approach

- Tip = **servRatio*(.2/10*(service)+.05) + (1-servRatio)*(.2/10*(food)+0.05);** **servRatio = 80%**



- Seems too linear. Want 15% tip in general and deviation only for exceptionally good or bad service.

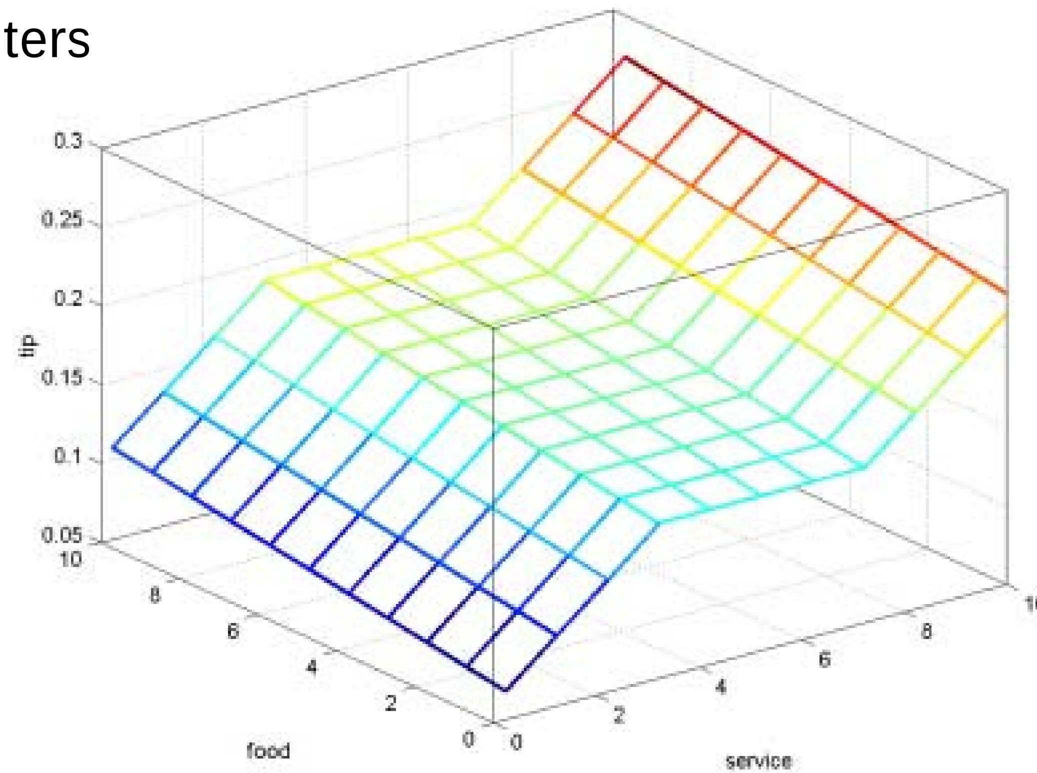
Tipping example: The non-fuzzy approach

```
if service < 3,
  tip(f+1,s+1) = servRatio*(.1/3*(s)+.05) + ...
                 (1-servRatio)*(.2/10*(f)+0.05);
elseif s < 7,
  tip(f+1,s+1) = servRatio*(.15) + ...
                 (1-servRatio)*(.2/10*(f)+0.05);
else,
  tip(f+1,s+1) = servRatio*(.1/3*(s-7)+.15) + ...
                 (1-servRatio)*(.2/10*(f)+0.05);
end;
```

Tipping example: The non-fuzzy approach

Nice plot but

- ‘Complicated’ function
- Not easy to modify
- Not intuitive
- Many hard-coded parameters
- Not easy to understand



Tipping problem: the fuzzy approach

What we want to express is:

1. *If service is poor then tip is cheap*
2. *If service is good the tip is average*
3. *If service is excellent then tip is generous*
4. *If food is rancid then tip is cheap*
5. *If food is delicious then tip is generous*

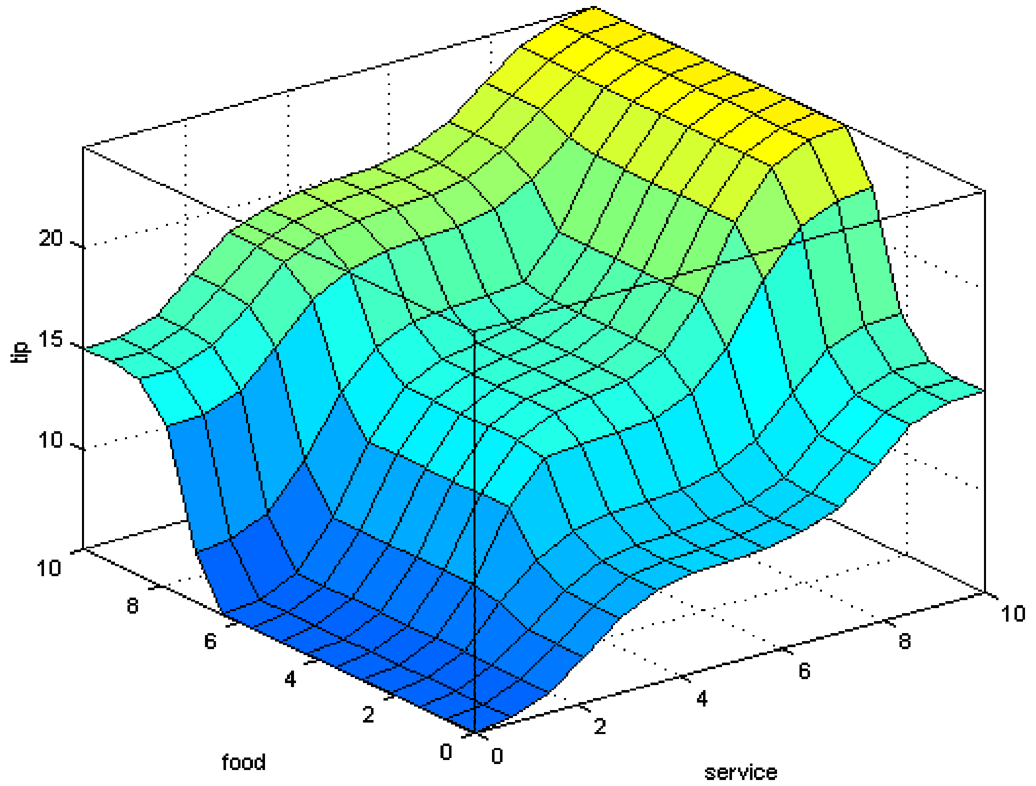
or

1. *If service is poor or the food is rancid then tip is cheap*
2. *If service is good then tip is average*
3. *If service is excellent or food is delicious then tip is generous*

We have just defined the rules for a fuzzy logic system.

Tipping problem: fuzzy solution

Decision function generated using the 3 rules.



Tipping problem: fuzzy solution

- Before we have a fuzzy solution we need to find out

a) how to define terms such as *poor, delicious*, *cheap, generous etc.*
b) how to combine terms using AND, OR and other connectives
c) how to combine all the rules into one final output

Fuzzy sets

- **Boolean / Crisp set A** is a mapping for the elements of S to the set {0, 1}, i.e., A: S → {0, 1}
- ***Characteristic function:***

$$\mu_A(x) = \begin{cases} 1 & \text{if x is an element of set A} \\ 0 & \text{if x is not an element of set A} \end{cases}$$

- **Fuzzy set F** is a mapping for the elements of S to the interval [0, 1], i.e., F: S → [0, 1]
- Characteristic function: $0 \leq \mu_F(x) \leq 1$
- 1 means full membership, 0 means no membership and anything in between, e.g., 0.5 is called **graded membership**

Example: Crisp set Tall

- Fuzzy sets and concepts are commonly used in natural language

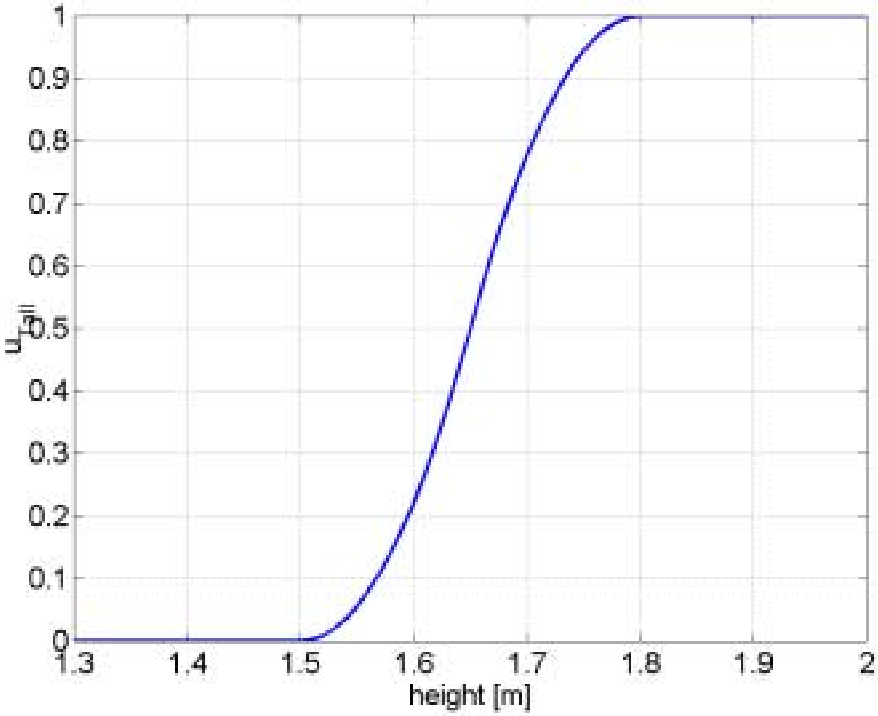
  *John is **tall***
  *Dan is **smart***
  *Alex is **happy***
  *The class is **hot***

- E.g., the crisp set ***Tall*** can be defined as $\{x \mid \text{height } x > 1.8 \text{ meters}\}$
  But what about a person with a height = 1.79 meters?
  What about 1.78 meters?
  …
  What about 1.52 meters?

Example: Fuzzy set Tall

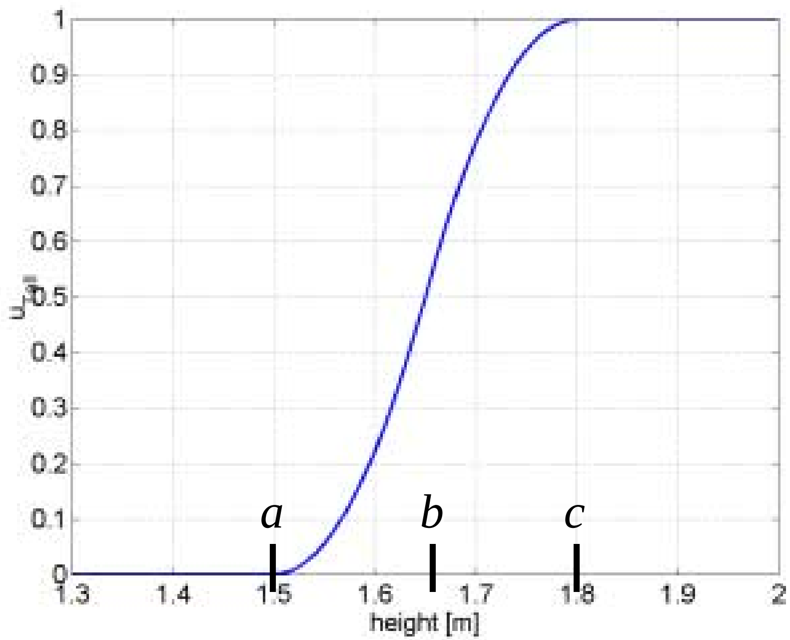
- In a fuzzy set a person with a height of 1.8 meters would be considered tall to a **high degree**
  A person with a height of 1.7 meters would be considered tall to a lesser degree etc.

- The function can change for basketball players, Danes, women, children etc.



Membership functions: S-function

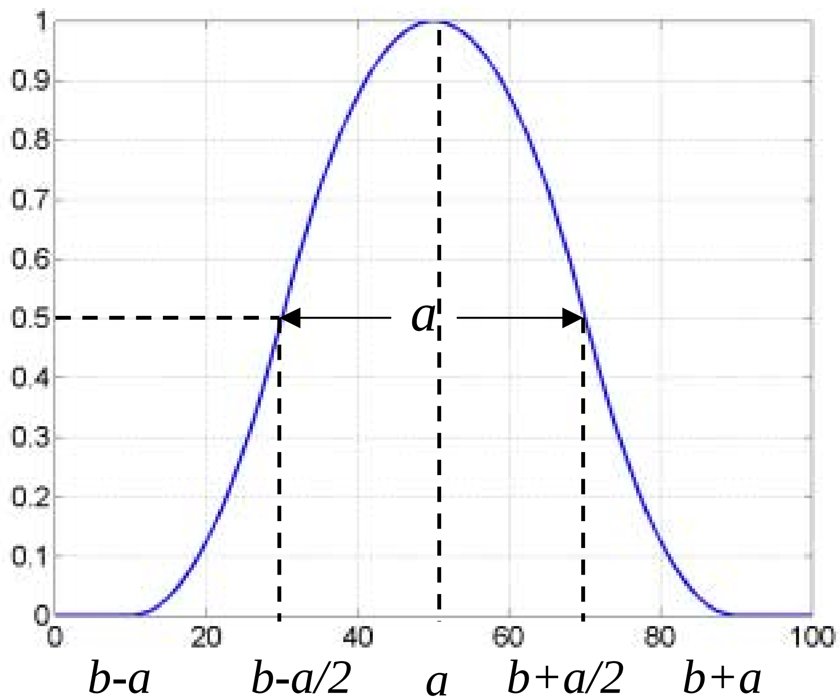
- The S-function can be used to define fuzzy sets
- $S(x, a, b, c) =$
  - 0 for $x \le a$
  - $2(x\text{-a/c-a})^2$ for $a \le x \le b$
  - $1 - 2(x\text{-}c/c\text{-}a)^2$ for $b \le x \le c$
  - 1 for $x \ge c$



Membership functions: Π–Function

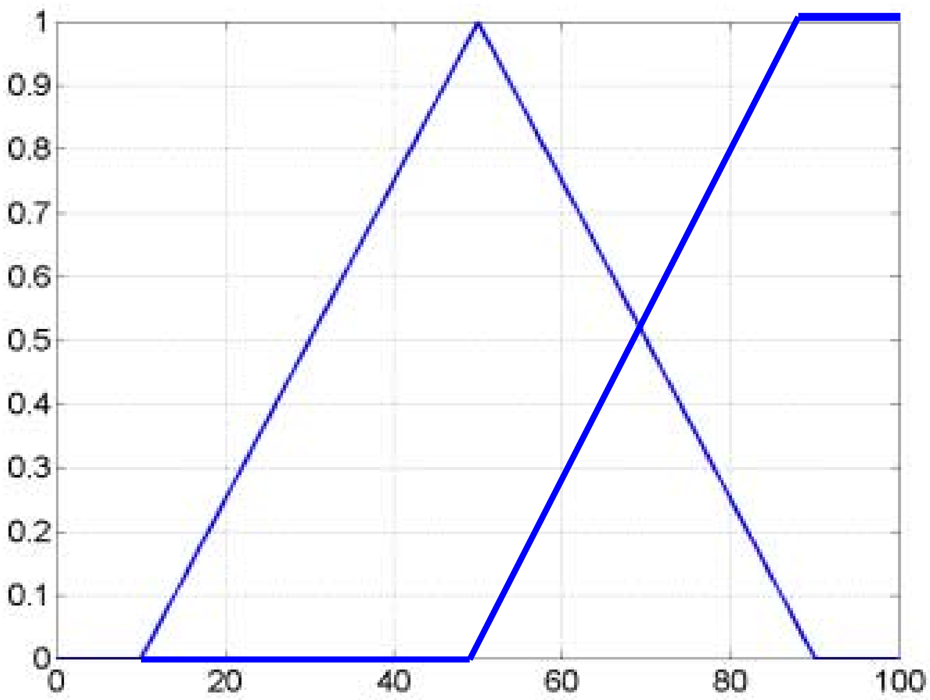
- $\Pi(x, a, b) =$
  - $S(x, b\text{-}a, b\text{-}a/2, b)$ for $x \le b$
  - $1 - S(x, b, b+a/2, a+b)$ for $x \ge b$

E.g., ***close*** (to a)



Simple membership functions

- Piecewise linear: triangular etc.
- Easier to represent and calculate ⇒ saves computation



Other representations of fuzzy sets

- A finite set of elements:

  $F = \mu_1/x_1 + \mu_2/x_2 + \ldots \mu_n/x_n$

  + means (Boolean) set union

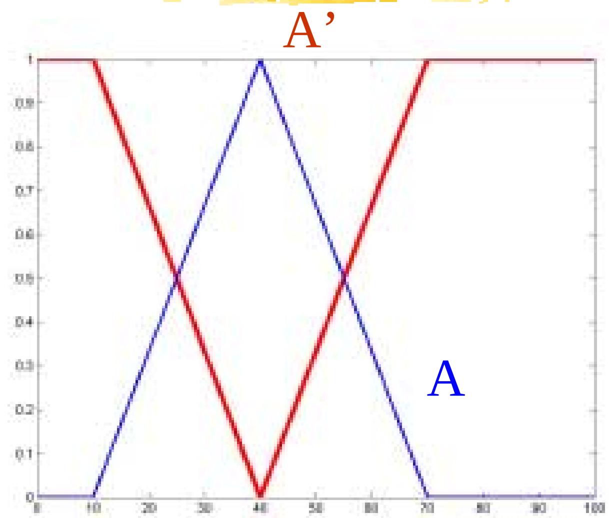
- For example:

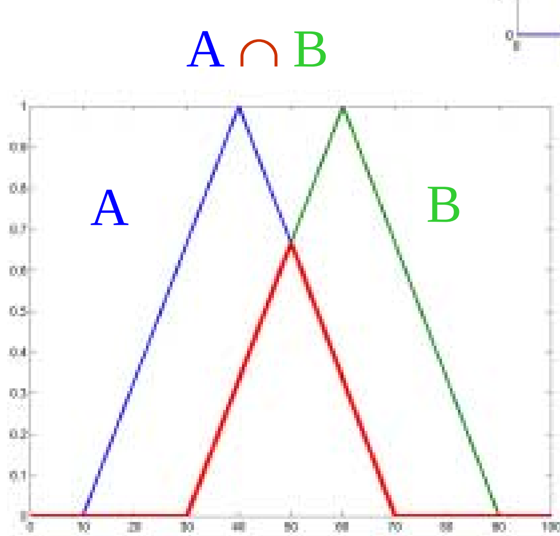
  TALL = {0/1.0, 0/1.2, 0/1.4, 0.2/1.6, 0.8/1.7, 1.0/1.8}

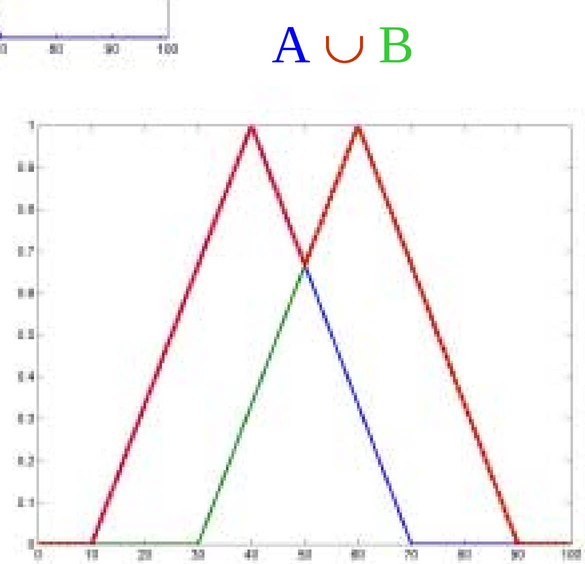
Fuzzy set operators

- Equality
  $A = B$
  $\mu_A(x) = \mu_B(x)$ for all $x \in X$
- Complement
  A'
  $\mu_{A'}(x) = 1 - \mu_A(x)$ for all $x \in X$
- Containment
  $A \subseteq B$
  $\mu_A(x) \le \mu_B(x)$ for all $x \in X$
- Union
  $A \cup B$
  $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$ for all $x \in X$
- Intersection
  $A \cap B$
  $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$ for all $x \in X$

Example fuzzy set operations



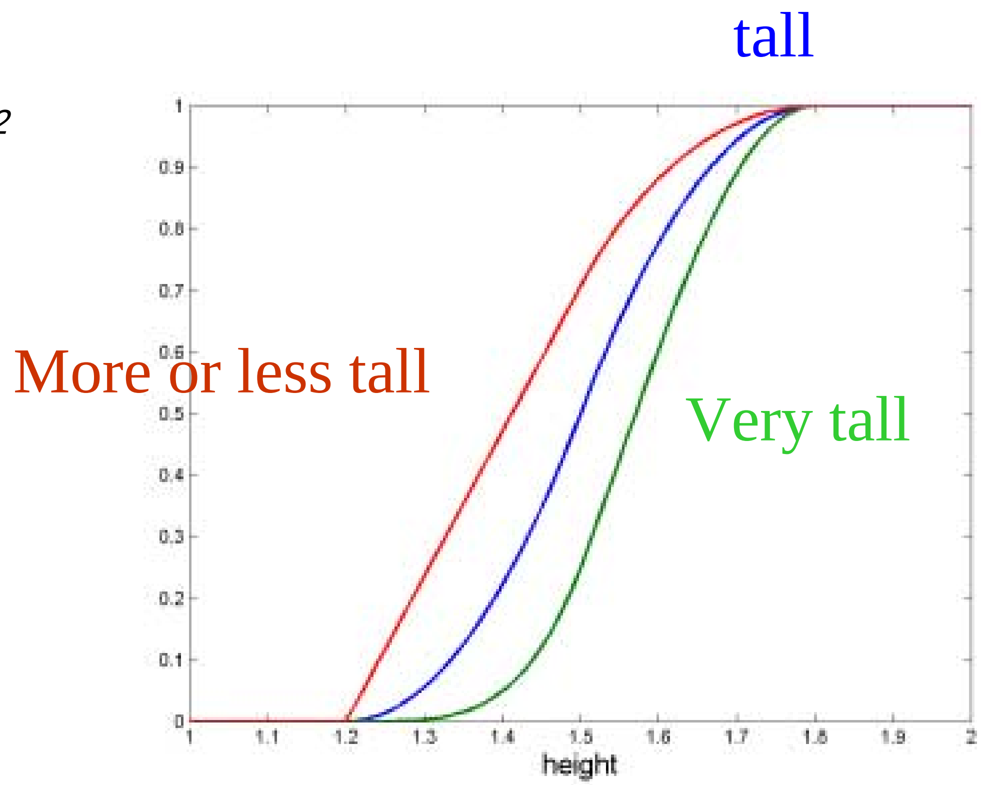




Linguistic Hedges

- Modifying the meaning of a fuzzy set using hedges such as *very, more or less, slightly, etc.*

- *Very F* = F^2
- *More or less F* = $F^{1/2}$
- *etc.*



Fuzzy relations

- A fuzzy relation for N sets is defined as an extension of the crisp relation to include the membership grade.

  R = $\{\mu_R(x_1, x_2, \ldots x_N)/(x_1, x_2, \ldots x_N) \mid x_i \in X, i=1, \ldots N\}$

  which associates the membership grade, μ_R , of each tuple.

- E.g.

  Friend = {0.9/(Manos, Nacho), 0.1/(Manos, Dan),
  0.8/(Alex, Mike), 0.3/(Alex, John)}

Fuzzy inference

- Fuzzy logical operations
- Fuzzy rules
- Fuzzification
- Implication
- Aggregation
- Defuzzification

Fuzzy logical operations

- AND, OR, NOT, etc.

- **NOT** A = A' = $1 - \mu_A(x)$
- A **AND** B = $A \cap B$ = $\min(\mu_A(x), \mu_B(x))$
- A **OR** B = $A \cup B$ = $\max(\mu_A(x), \mu_B(x))$

From the following truth tables it is seen that fuzzy logic is a **superset** of Boolean logic.

min(A,B)

A	B	A and B
0	0	0
0	1	0
1	0	0
1	1	1

max(A,B)

A	B	A or B
0	0	0
0	1	1
1	0	1
1	1	1

1-A

A	not A
0	1
1	0

If-Then Rules

- Use fuzzy sets and fuzzy operators as the **subjects** and **verbs** of fuzzy logic to form rules.

  if x is A then y is B

  where A and B are linguistic terms defined by fuzzy sets on the sets X and Y respectively.

  This reads

  if <u>x == A</u> then <u>y = B</u>

Evaluation of fuzzy rules

- In Boolean logic: $p \Rightarrow q$
  if p is true then q is true

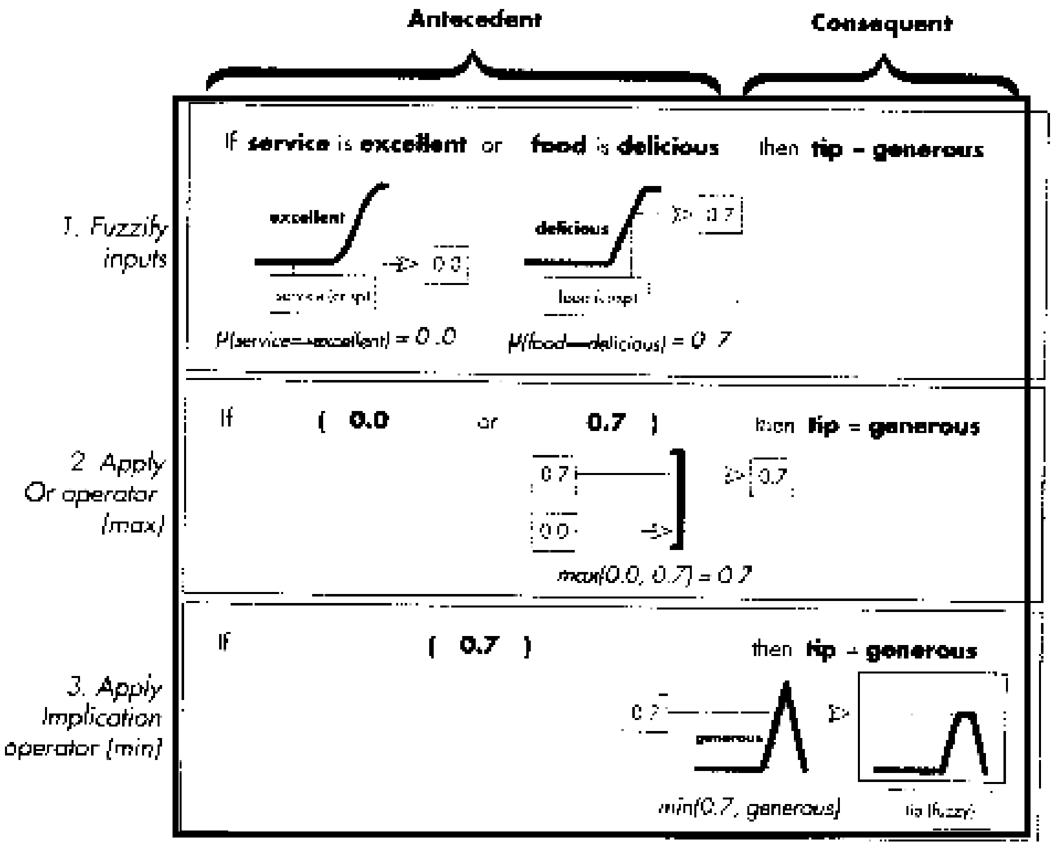
- In fuzzy logic: $p \Rightarrow q$
  if p is true to some degree then q is true to some degree.

  0.5p => 0.5q (partial premise implies partially)

- How?

Evaluation of fuzzy rules (cont'd)

- Apply **implication function** to the rule
- Most common way is to use min to "chop-off" the consequent (prod can be used to scale the consequent)



Summary: If-Then rules

1. Fuzzify inputs
   Determine the degree of membership for all terms in the premise. If there is one term then this is the degree of support for the consequence.

2. Apply fuzzy operator
   If there are multiple parts, apply logical operators to determine the degree of support for the rule.

3. Apply implication method
   Use degree of support for rule to shape output fuzzy set of the consequence.

How do we then combine several rules?

Multiple rules

- We aggregate the outputs into a single fuzzy set which combines their decisions.
- The input to aggregation is the list of truncated fuzzy sets and the output is a single fuzzy set for each variable.
- **Aggregation rules**: max, sum, etc.
- As long as it is commutative then the order of rule exec is irrelevant.

max-min rule of composition

- Given N observations E_i over X and hypothesis H_i over Y we have N rules:

  if E_1 then H_1
  if E_2 then H_2

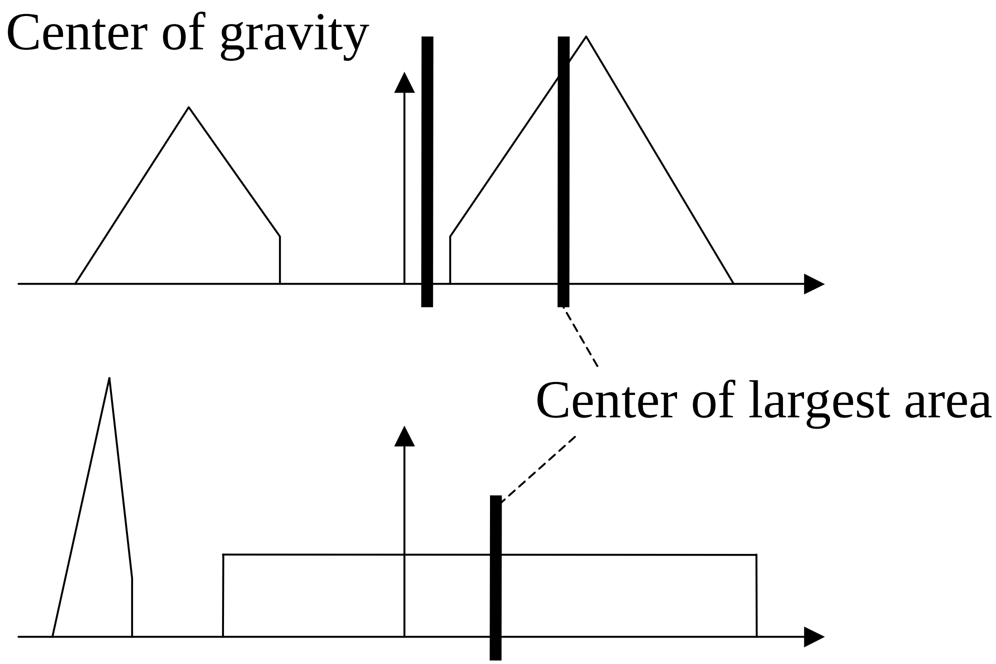
  if E_N then H_N

- $\mu_H = \max[\min(\mu_{E1}), \min(\mu_{E2}), \ldots \min(\mu_{EN})]$

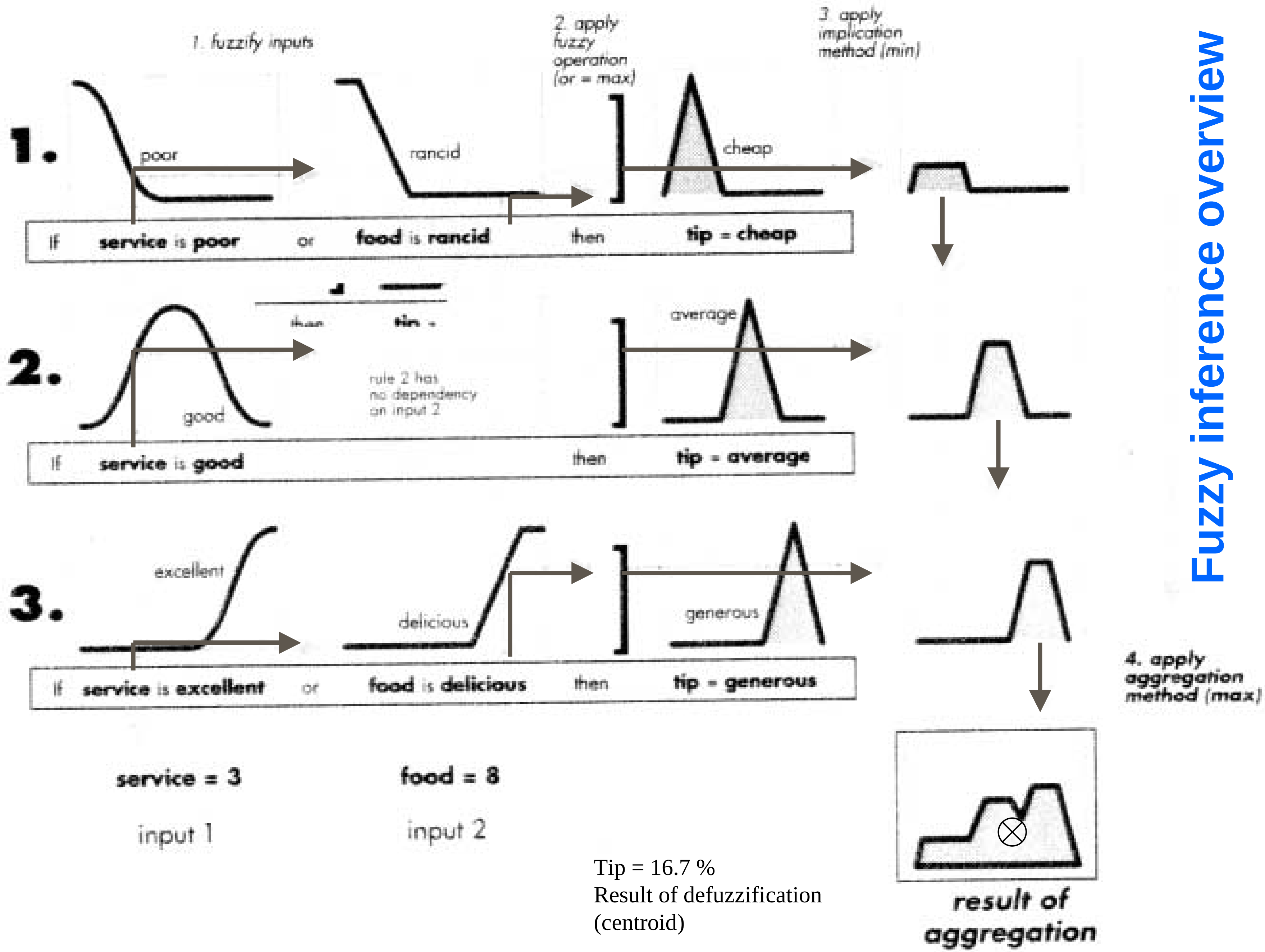
Defuzzify the output

- Take a fuzzy set and produce a single crisp number that represents the set.
- Practical when making a decision, taking an action etc.

$$I = \frac{\Sigma \mu_I x}{\Sigma \mu_I}$$



Fuzzy inference overview



Tip = 16.7 %
Result of defuzzification
(centroid)

Limitations of fuzzy logic

- How to determine the membership functions? Usually requires fine-tuning of parameters
- Defuzzification can produce undesired results

Fuzzy tools and shells

- Matlab’s Fuzzy Toolbox
- FuzzyClips
- Etc.