This time: Fuzzy Logic and Fuzzy Inference

- Why use fuzzy logic?
- Tipping example
- Fuzzy set theory
- Fuzzy inference
What is fuzzy logic?

- A super set of Boolean logic
- Builds upon fuzzy set theory
- Graded truth. Truth values between True and False. Not everything is either/or, true/false, black/white, on/off etc.
- Grades of membership. Class of tall men, class of far cities, class of expensive things, etc.
- Lotfi Zadeh, UC/Berkely 1965. Introduced FL to model uncertainty in natural language. Tall, far, nice, large, hot, ...
- Reasoning using linguistic terms. Natural to express expert knowledge.
  - *If the weather is cold then wear warm clothing*
Why use fuzzy logic?

Pros:
• Conceptually easy to understand w/ “natural” maths
• Tolerant of imprecise data
• Universal approximation: can model arbitrary nonlinear functions
• Intuitive
• Based on linguistic terms
• Convenient way to express expert and common sense knowledge

Cons:
• Not a cure-all
• Crisp/precise models can be more efficient and even convenient
• Other approaches might be formally verified to work
Tipping example

- **The Basic Tipping Problem:** Given a number between 0 and 10 that represents the quality of service at a restaurant what should the tip be?

  Cultural footnote: An average tip for a meal in the U.S. is 15%, which may vary depending on the quality of the service provided.
Tipping example: The non-fuzzy approach

- Tip = 15% of total bill

- What about quality of service?
Tipping example: The non-fuzzy approach

- Tip = linearly proportional to service from 5% to 25%
  \[ \text{tip} = \frac{0.20}{10}\times\text{service} + 0.05 \]

- What about quality of the food?
Tipping example: Extended

- **The Extended Tipping Problem:** Given a number between 0 and 10 that represents the quality of service and the quality of the food, at a restaurant, what should the tip be?

How will this affect our tipping formula?
Tipping example: The non-fuzzy approach

- Tip = 0.20/20*(service+food)+0.05

- We want service to be more important than food quality. E.g., 80% for service and 20% for food.
Tipping example: The non-fuzzy approach

- Tip = servRatio*(.2/10*(service)+.05) +
  (1-servRatio)*(.2/10*(food)+0.05);
  servRatio = 80%

- Seems too linear. Want 15% tip in general and deviation only for exceptionally good or bad service.
Tipping example: The non-fuzzy approach

if service < 3,
    \[ \text{tip}(f+1,s+1) = \text{servRatio} \times (0.1/3 \times (s) + 0.05) + \ldots + (1-\text{servRatio}) \times (0.2/10 \times (f) + 0.05) \];
elseif \( s < 7 \),
    \[ \text{tip}(f+1,s+1) = \text{servRatio} \times (0.15) + \ldots + (1-\text{servRatio}) \times (0.2/10 \times (f) + 0.05) \];
else,
    \[ \text{tip}(f+1,s+1) = \text{servRatio} \times (0.1/3 \times (s-7) + 0.15) + \ldots + (1-\text{servRatio}) \times (0.2/10 \times (f) + 0.05) \];
end;
Tipping example: The non-fuzzy approach

Nice plot but
- ‘Complicated’ function
- Not easy to modify
- Not intuitive
- Many hard-coded parameters
- Not easy to understand
Tipping problem: the fuzzy approach

What we want to express is:

1. If service is poor then tip is cheap
2. If service is good then tip is average
3. If service is excellent then tip is generous
4. If food is rancid then tip is cheap
5. If food is delicious then tip is generous

or

1. If service is poor or the food is rancid then tip is cheap
2. If service is good then tip is average
3. If service is excellent or food is delicious then tip is generous

We have just defined the rules for a fuzzy logic system.
Tipping problem: fuzzy solution

Decision function generated using the 3 rules.
Tipping problem: fuzzy solution

• Before we have a fuzzy solution we need to find out

a) how to define terms such as poor, delicious, cheap, generous etc.
b) how to combine terms using AND, OR and other connectives
c) how to combine all the rules into one final output
Fuzzy sets

- **Boolean/ Crisp set** $A$ is a mapping for the elements of $S$ to the set $\{0, 1\}$, i.e., $A: S \rightarrow \{0, 1\}$
- **Characteristic function:**
  
  $\mu_A(x) = \begin{cases} 
  1 & \text{if } x \text{ is an element of set } A \\
  0 & \text{if } x \text{ is not an element of set } A 
  \end{cases}$

- **Fuzzy set** $F$ is a mapping for the elements of $S$ to the interval $[0, 1]$, i.e., $F: S \rightarrow [0, 1]$
- Characteristic function: $0 \leq \mu_F(x) \leq 1$
- 1 means full membership, 0 means no membership and anything in between, e.g., 0.5 is called **graded membership**
Example: Crisp set Tall

- Fuzzy sets and concepts are commonly used in natural language

*Example: Crisp set Tall*

- John is *tall*
  - *Dan is smart*
  - *Alex is happy*
  - *The class is hot*

- E.g., the crisp set *Tall* can be defined as \( \{ x \mid \text{height } x > 1.8 \text{ meters} \} \)
  - But what about a person with a height = 1.79 meters?
  - What about 1.78 meters?
  - ...  
  - What about 1.52 meters?
Example: Fuzzy set Tall

- In a fuzzy set a person with a height of 1.8 meters would be considered tall to a **high degree**. A person with a height of 1.7 meters would be considered tall to a lesser degree etc.

- The function can change for basketball players, Danes, women, children etc.
Membership functions: S-function

- The S-function can be used to define fuzzy sets
- \( S(x, a, b, c) = \)
  - 0 for \( x \leq a \)
  - \( 2(x-a/c-a)^2 \) for \( a \leq x \leq b \)
  - \( 1 - 2(x-c/c-a)^2 \) for \( b \leq x \leq c \)
  - 1 for \( x \geq c \)
Membership functions: Π–Function

- \( \Pi(x, a, b) = \)
  - \( S(x, b-a, b-a/2, b) \) for \( x \leq b \)
  - \( 1 - S(x, b, b+a/2, a+b) \) for \( x \geq b \)

E.g., close (to a)
Simple membership functions

- Piecewise linear: triangular etc.
- Easier to represent and calculate $\Rightarrow$ saves computation
Fuzzy Sets

Membership Grade

\[ \mu \]

Cold, Mild, Warm

°F
Observation

![Graph showing temperature intervals for Cold, Mild, and Warm conditions with associated degrees Fahrenheit and membership values.]

- Cold: 
  - Membership value: 0.85

- Mild: 
  - Membership value: 1

- Warm: 
  - Membership value: 0.14

- Degree Fahrenheit: 38°
Other representations of fuzzy sets

- A finite set of elements:

  \[ F = \mu_1/x_1 + \mu_2/x_2 + \ldots + \mu_n/x_n \]

  + means (Boolean) set union

- For example:

  \[ \text{TALL} = \{0/1.0, 0/1.2, 0/1.4, 0.2/1.6, 0.8/1.7, 1.0/1.8\} \]
Fuzzy set operators

- **Equality**
  \[ A = B \]
  \[ \mu_A(x) = \mu_B(x) \quad \text{for all } x \in X \]

- **Complement**
  \[ A' \]
  \[ \mu_{A'}(x) = 1 - \mu_A(x) \quad \text{for all } x \in X \]

- **Containment**
  \[ A \subseteq B \]
  \[ \mu_A(x) \leq \mu_B(x) \quad \text{for all } x \in X \]

- **Union**
  \[ A \cup B \]
  \[ \mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x)) \quad \text{for all } x \in X \]

- **Intersection**
  \[ A \cap B \]
  \[ \mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x)) \quad \text{for all } x \in X \]
Example fuzzy set operations

\[ A \cap A' \]

\[ A \cup B \]

\[ A' \cap B \]

\[ A \cup B \]
Linguistic Hedges

- Modifying the meaning of a fuzzy set using hedges such as very, more or less, slightly, etc.

- Very \( F = F^2 \)
- More or less \( F = F^{1/2} \)
- etc.
Fuzzy relations

- A fuzzy relation for N sets is defined as an extension of the crisp relation to include the membership grade.

\[ R = \{ \mu_R(x_1, x_2, \ldots x_N)/(x_1, x_2, \ldots x_N) \mid x_i \in X, i=1, \ldots N\} \]

which associates the membership grade, \( \mu_R \), of each tuple.

- E.g.

\[ \text{Friend} = \{0.9/(\text{Manos, Nacho}), 0.1/(\text{Manos, Dan}), 0.8/(\text{Alex, Mike}), 0.3/(\text{Alex, John})\} \]
Fuzzy inference

- Fuzzy logical operations
- Fuzzy rules
- Fuzzification
- Implication
- Aggregation
- Defuzzification
# Fuzzy logical operations

- **AND, OR, NOT, etc.**

- **NOT** $A = A' = 1 - \mu_A(x)$
- **A AND B** $= A \cap B = \min(\mu_A(x), \mu_B(x))$
- **A OR B** $= A \cup B = \max(\mu_A(x), \mu_B(x))$

From the following truth tables it is seen that fuzzy logic is a **superset** of Boolean logic.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>A and B</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>A or B</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
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<td>1</td>
<td>0</td>
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</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>A</th>
<th>not A</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
If-Then Rules

- Use fuzzy sets and fuzzy operators as the subjects and verbs of fuzzy logic to form rules.

\[
\text{if } x \text{ is } A \text{ then } y \text{ is } B
\]

where A and B are linguistic terms defined by fuzzy sets on the sets X and Y respectively.

This reads

\[
\text{if } x == A \text{ then } y = B
\]
Example:

- IF height is Tall THEN weight is Heavy
Example

- If it is hot, turn on the air conditioner
  - Determine if the current temp. belongs to the hot fuzzy set
  - If so, then turn on the air conditioner until it goes to the warm fuzzy set
Evaluation of fuzzy rules

- In Boolean logic: \( p \Rightarrow q \)
  if \( p \) is true then \( q \) is true

- In fuzzy logic: \( p \Rightarrow q \)
  if \( p \) is true to some degree then \( q \) is true to some degree.

\[ 0.5p \Rightarrow 0.5q \] (partial premise implies partially)

- How?
A Very Simple Example

- Fuzzification

![Graph showing fuzzification](image)

- Cold
- Mild
- Warm
A Very Simple Example

- Inferencing

\[ \mu \]

\[
\begin{array}{c}
0 \\
1 \\
30 \\
38^\circ \\
60 \\
\end{array}
\]

Cold Mild Warm

°F
A Very Simple Example

- Rule Evaluation
- If the temp. is mild, then no action

![Diagram showing temperature ranges and corresponding actions for heater and air conditioner.](image)
A Very Simple Example

- Rule Evaluation

\[ \begin{array}{c}
\text{F} \\
\text{μ}
\end{array} \]

\[ \begin{array}{c}
0 \quad 30 \quad 60 \quad 38° \quad 0
\end{array} \]

- Mild
- No Action

Air conditioner
Center of gravity
Full Example:

Membership Grade

μ

Cold

Mild

Warm

°C

μ

1

0

30

60

°F
Fuzzification

The diagram illustrates the fuzzification process with temperature ranging from 30°F to 60°F. The temperature range is divided into three categories: Cold, Mild, and Warm. The graph shows the membership functions for each category. The cold category has a membership value of 0.85 at 38°F, the mild category has a membership value of 0.14 at 38°F, and the warm category has a membership value of 1 at 60°F. This representation helps in understanding the degree of membership of a temperature within each category.
Set Operators: AND/OR

(AND/OR) Min or Max: Depending on the rule, select the value to decide the results of the rule.
Set Operators: AND/OR

IF temp is Mild THEN No Action
This is a single item rule. No min/max applied
Set Operators: AND/OR

- If you have a sick kid at home and it is cold for him.
Set Operators: Max value

- **Cold**
- **Mild**
- **Warm**

![Graph showing temperatures and membership functions]

At 38°F, it is mild, no need for heating up.
Set Operators: Max value

It is mild, no need for heating up

CS 460, Sessions 22-23
Question

Give an example to make it depending on two inputs.
Fuzzy Rules

• Example: “If our distance to the car in front is small, and the distance is decreasing slowly, then decelerate quite hard”
  • Fuzzy variables in blue
  • Fuzzy sets in red

• QUESTION: Given the distance and the change in the distance, what acceleration should we select?
Fuzzification: Set Definitions

- For **distance**:
  - v. small
  - small
  - perfect
  - big
  - v. big

- For **acceleration**:
  - brake
  - slow
  - present
  - fast
  - fastest

- For **Delta (distance change)**:
  - <<
  - <
  - =
  - >
  - >>
Fuzzification: Instance

- Distance could be considered small or perfect
- Delta could be stable or growing
- What acceleration?
Fuzzification: Instance

IF distance is Small THEN Slow Down
Distance is small, then you slow down.
Question: What is the weight to slow down?
Distance is small, then you slow down.
Fuzzification: Instance

IF change in distance is = THEN Keep the speed

\[ \begin{array}{c}
<< & < & = & > & >> \\
\end{array} \]

\[ \begin{array}{c}
0.75 \\
0.23 \\
\end{array} \]
Rule Evaluation

Distance is not growing, then keep present acceleration

\[ \text{distance} = 0.75 \]

\( \text{delta} \quad \text{acceleration} \)

brake \quad slow \quad present \quad fast \quad fastest

Distance is not growing, then keep present acceleration
Rule Evaluation

Distance is not growing, then keep present acceleration

\[ \text{delta} \quad \text{acceleration} \]

Distance is not growing, then keep present acceleration
Rule Aggregation

From distance
From delta (distance change)
Rule Aggregation

So what should we do? Current acceleration or slow down?
Defuzzification

So what should we do? Present acceleration or slow down?
Rule Aggregation: Another case

- Convert our belief into action
  - For each rule, clip action fuzzy set by belief in rule
Rule Aggregation: Another case

• Convert our belief into action
  • For each rule, clip action fuzzy set by belief in rule
Matching for Example

- Relevant rules are:
  - If distance is small and delta is growing, maintain speed
  - If distance is small and delta is stable, slow down
  - If distance is perfect and delta is growing, speed up
  - If distance is perfect and delta is stable, maintain speed
Matching for Example

- For first rule, distance is small has 0.75 truth, and delta is growing has 0.3 truth
  - So the truth of the **and** is 0.3
- Other rule strengths are 0.6, 0.1 and 0.1
AND/OR Example

- IF Distance Small AND change in distance negative THEN high deceleration
AND/OR Example

- IF Distance Small AND change in distance = THEN slow deceleration
AND/OR Example

- IF Distance Small AND change in distance = THEN slow deceleration
Scaling vs. Clipping

Acceleration

Present

Slow
Evaluation of fuzzy rules (cont’d)

- Apply **implication function** to the rule
- Most common way is to use `min` to “chop-off” the consequent (prod can be used to scale the consequent)
Summary: If-Then rules

1. Fuzzify inputs:
   Determine the degree of membership for all terms in the premise.
   If there is one term then this is the degree of support for the consequence.

2. Apply fuzzy operator:
   If there are multiple parts, apply logical operators to determine the degree of support for the rule.
Summary: If-Then rules

3. Apply implication method:
Use degree of support for rule to shape output fuzzy set of the consequence.

How do we then combine several rules?
Multiple rules

- We aggregate the outputs into a single fuzzy set which combines their decisions.
- The input to aggregation is the list of truncated fuzzy sets and the output is a single fuzzy set for each variable.
- **Aggregation rules**: max, sum, etc.
- As long as it is commutative then the order of rule exec is irrelevant.
max-min rule of composition

- Given $N$ observations $E_i$ over $X$ and hypothesis $H_i$ over $Y$ we have $N$ rules:

  if $E_1$ then $H_1$
  if $E_2$ then $H_2$

  if $E_N$ then $H_N$

- $\mu_H = \max[\min(\mu_{E_1}), \min(\mu_{E_2}), \ldots \min(\mu_{E_N})]$
Defuzzify the output

- Take a fuzzy set and produce a single crisp number that represents the set.
- Practical when making a decision, taking an action etc.

$$I = \frac{\sum \mu_i \times x}{\sum \mu_i}$$

Center of gravity

Center of largest area
Fuzzy inference overview

1. fuzzify inputs

- If service is poor or food is rancid, then tip = cheap

2. apply fuzzy operation (or = max)

- Rule 2 has no dependency on input 2

- If service is good, then tip = average

3. apply implication method (min)

- If service is excellent or food is delicious, then tip = generous

4. apply aggregation method (max)

- Input 1: service = 3
- Input 2: food = 8

Tip = 16.7%

Result of defuzzification (centroid)
Limitations of fuzzy logic

- How to determine the membership functions? Usually requires fine-tuning of parameters

- Defuzzification can produce undesired results
Fuzzy tools and shells

- Matlab’s Fuzzy Toolbox
- FuzzyClips
- Etc.
Camcorder Example

• Stabilizer operates by attempting to identify the subject versus the background. Using this, we can determine whether it is the subject and/or background that is moving, or if it is the holder of the camcorder that is moving.
Camcorder Example

- One method is to use a set of input points in a grid and poll those points twice per second.
- Between pollings, the camcorder deduces which direction the objects have shifted....
Camcorder Example

• If the subject moves, then the camcorder detects a shift among points in a localized region. If this region is somewhere in the lower center of the shot, then the chances that it is a subject-move is even greater.
However, if it appears that a significant amount of the screen is shifting yet a localized region is standing still, then the camcorder can deduce that the background is moving while the subject is not.
Camcorder Example

- If it appears that the entire picture has shifted and that there is no distinction of subject or background, the camcorder can identify this and deduce that the camera-holder’s hand has shifted. The camcorder can then compensate for the shift.
Camcorder Example

• The fuzzy logic would work as follows:
  • Fuzzification: The fuzzy set could be: red, orange, yellow, ..., purple, black, and white. Each pixel is identified as having a degree of each of these colors based on the levels of red, green, and blue detected.
  • Inference: First layer of rules deduce where the shifts occur among single pixels. Second layer of rules clump together like shifts into shifted regions.
  • Composition: Based on the collected evidence, deduce overall shift of camcorder (slight up-down, slight left-right, ....)
  • Defuzzification: Translate the overall shift of camcorder into compensatory action (slight up-down: shift picture up 1 pixel...)