Administrativia

• Assignment 1 due tuesday 9/23/2003 BEFORE midnight

• Midterm exam 10/09/2003
Last time: search strategies

**Uninformed:** Use only information available in the problem formulation
- Breadth-first
- Uniform-cost
- Depth-first
- Depth-limited
- Iterative deepening

**Informed:** Use heuristics to guide the search
- Best first:
- Greedy search – queue first nodes that maximize heuristic “desirability” based on estimated path cost from current node to goal;
- A* search – queue first nodes that maximize sum of path cost so far and estimated path cost to goal.
- Iterative improvement – keep no memory of path; work on a single current state and iteratively improve its “value.”
- Hill climbing – select as new current state the successor state which maximizes value.
- Simulated annealing – refinement on hill climbing by which “bad moves” are permitted, but with decreasing size and frequency. Will find global extremum.
Exercise: Search Algorithms

The following figure shows a portion of a partially expanded search tree. Each arc between nodes is labeled with the cost of the corresponding operator, and the leaves are labeled with the value of the heuristic function, $h$.

Which node (use the node’s letter) will be expanded next by each of the following search algorithms?

(a) Depth-first search  
(b) Breadth-first search  
(c) Uniform-cost search  
(d) Greedy search  
(e) A* search
# Depth-first search

Node queue: *initialization*

<table>
<thead>
<tr>
<th>#</th>
<th>state</th>
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<th>path cost</th>
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## Depth-first search

Node queue: add successors to queue front; empty queue from top

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Depth-first search

Node queue: add successors to queue front; empty queue from top

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**Breadth-first search**

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Breadth-first search

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Breadth-first search

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Uniform-cost search

Node queue: initialization

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Uniform-cost search

Node queue: add successors to queue so that entire queue is sorted by path cost so far; empty queue from top

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Uniform-cost search

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</table>
# Uniform-cost search

Node queue: *add successors to queue so that entire queue is sorted by path cost so far; empty queue from top*

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Exercise: Search Algorithms

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Greedy search

Node queue: initialization

<table>
<thead>
<tr>
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**Greedy search**

Node queue:  
*Add successors to queue, sorted by cost to goal.*

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Greedy search

Node queue: Add successors to queue, sorted by cost to goal.

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**Greedy search**

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### A* search

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<td>A</td>
<td>0</td>
<td>0</td>
<td>20</td>
<td>20</td>
<td>--</td>
<td></td>
</tr>
</tbody>
</table>
A* search

Node queue: Add successors to queue, sorted by total cost.

<table>
<thead>
<tr>
<th>#</th>
<th>state</th>
<th>depth</th>
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<tr>
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<td>1</td>
</tr>
<tr>
<td>4</td>
<td>C</td>
<td>1</td>
<td>19</td>
<td>18</td>
<td>37</td>
<td>1</td>
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Sort key
A* search

Node queue: Add successors to queue front, sorted by total cost.

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### A* search

Node queue: **Add successors to queue front, sorted by total cost.**

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</table>
Exercise: Search Algorithms

The following figure shows a portion of a partially expanded search tree. Each arc between nodes is labeled with the cost of the corresponding operator, and the leaves are labeled with the value of the heuristic function, $h$.

Which node (use the node’s letter) will be expanded next by each of the following search algorithms?

(a) Depth-first search
(b) Breadth-first search
(c) Uniform-cost search
(d) Greedy search
(e) A* search

CS 561, Sessions 8-9
Last time: Simulated annealing algorithm

- Idea: Escape local extrema by allowing “bad moves,” but gradually decrease their size and frequency.

```
function SIMULATED-ANNEALING(problem, schedule) returns a solution state
  inputs: problem, a problem
           schedule, a mapping from time to “temperature”
  local variables: current, a node
                   next, a node
                   T, a “temperature” controlling the probability of downward steps
  current ← MAKE-NODE(INITIAL-STATE[problem])
  for t ← 1 to ∞ do
    T ← schedule[t]
    if T=0 then return current
    next ← a randomly selected successor of current
    ΔE ← VALUE[next] - VALUE[current]
    if ΔE > 0 then current ← next
    else current ← next only with probability $e^{ΔE/T}$
```

Note: goal here is to maximize E.
Last time: Simulated annealing algorithm

- Idea: Escape local extrema by allowing “bad moves,” but gradually decrease their size and frequency.

```plaintext
function Simulated-Annealing(problem, schedule) returns a solution state
inputs: problem, a problem
        schedule, a mapping from time to “temperature”
local variables: current, a node
                next, a node
                T, a “temperature” controlling the probability of downward steps

current ← Make-Node(Initial-State[problem])
for t ← 1 to ∞ do
    T ← schedule[t]
    if T=0 then return current
next ← a randomly selected successor of current
ΔE ← Value[next] - Value[current]
if ΔE < 0 then current ← next
else current ← next only with probability e^ΔE/T
```

Algorithm when goal is to minimize E.
This time: Outline

- **Game playing**
  - The minimax algorithm
  - Resource limitations
  - alpha-beta pruning
  - Elements of chance
What kind of games?

- **Abstraction**: To describe a game we must capture every relevant aspect of the game. Such as:
  - Chess
  - Tic-tac-toe
  - ...

- **Accessible environments**: Such games are characterized by perfect information

- **Search**: game-playing then consists of a search through possible game positions

- **Unpredictable opponent**: introduces uncertainty thus game-playing must deal with contingency problems
Searching for the next move

- **Complexity**: many games have a huge search space
  - **Chess**: \( b = 35, \ m=100 \Rightarrow \text{nodes} = 35^{100} \)
    - if each node takes about 1 ns to explore
    - then each move will take about \( 10^{50} \) millennia to calculate.

- **Resource (e.g., time, memory) limit**: optimal solution not feasible/possible, thus must approximate

1. **Pruning**: makes the search more efficient by discarding portions of the search tree that cannot improve quality result.

2. **Evaluation functions**: heuristics to evaluate utility of a state without exhaustive search.
Two-player games

- A game formulated as a search problem:
  - Initial state: ?
  - Operators: ?
  - Terminal state: ?
  - Utility function: ?
Two-player games

- A game formulated as a search problem:

  - Initial state: board position and turn
  - Operators: definition of legal moves
  - Terminal state: conditions for when game is over
  - Utility function: a numeric value that describes the outcome of the game. E.g., -1, 0, 1 for loss, draw, win. (AKA payoff function)
Game vs. search problem

“Unpredictable” opponent ⇒ solution is a contingency plan

Time limits ⇒ unlikely to find goal, must approximate

Plan of attack:

- algorithm for perfect play (Von Neumann, 1944)
- finite horizon, approximate evaluation (Zuse, 1945; Shannon, 1950; Samuel, 1952–57)
- pruning to reduce costs (McCarthy, 1956)
Example: Tic-Tac-Toe
Type of games

<table>
<thead>
<tr>
<th></th>
<th>deterministic</th>
<th>chance</th>
</tr>
</thead>
<tbody>
<tr>
<td>perfect info</td>
<td>chess, checkers, go,</td>
<td>backgammon</td>
</tr>
<tr>
<td></td>
<td>othello</td>
<td>monopoly</td>
</tr>
<tr>
<td>imperfect info</td>
<td></td>
<td>bridge, poker, scrabble</td>
</tr>
<tr>
<td></td>
<td></td>
<td>nuclear war</td>
</tr>
</tbody>
</table>
Type of games

- **Deterministic**
  - perfect information: chess, checkers, go, othello
  - imperfect information: backgammon, monopoly

- **Chance**
  - backgammon
  - monopoly
  - bridge, poker, scrabble
  - nuclear war

CS 561, Sessions 8-9
The minimax algorithm

- Perfect play for deterministic environments with perfect information

- **Basic idea:** choose move with highest minimax value
  \[= \text{best achievable payoff against best play}\]

- **Algorithm:**
  1. Generate game tree completely
  2. Determine utility of each terminal state
  3. Propagate the utility values upward in the tree by applying MIN and MAX operators on the nodes in the current level
  4. At the root node use minimax decision to select the move with the max (of the min) utility value

- Steps 2 and 3 in the algorithm assume that the opponent will play perfectly.
Generate Game Tree
Generate Game Tree
Generate Game Tree
Generate Game Tree

1 ply

1 move

CS 561, Sessions 8-9
A subtree
What is a good move?
Minimax

- Minimize opponent’s chance
- Maximize your chance
Minimax

- Minimize opponent’s chance
- Maximize your chance
Minimax

- Minimize opponent’s chance
- Maximize your chance

Diagram:

```
      3
     / \  \/
    3   2  2
   /   |  |
  3   12  8  2  4  6  14  5  2
```

Max

Min
Minimax

- Minimize opponent’s chance
- Maximize your chance
minimax = maximum of the minimum

Diagram:

MAX

MIN

1\text{st} ply

2\text{nd} ply
Minimax: Recursive implementation

\[
\text{function } \text{MINIMAX-DECISION}(game) \text{ returns an operator} \\
\text{for each } op \text{ in OPERATORS}[game] \text{ do} \\
\quad \text{Value}[op] \leftarrow \text{MINIMAX-VALUE}(\text{APPLY}(op, game), game) \\
\text{end} \\
\text{return the } op \text{ with the highest } \text{Value}[op]
\]

\[
\text{function } \text{MINIMAX-VALUE}(state, game) \text{ returns a utility value} \\
\text{if } \text{TERMINAL-TEST}[game](state) \text{ then} \\
\quad \text{return } \text{UTILITY}[game](state) \\
\text{else if } \text{MAX is to move in } state \text{ then} \\
\quad \text{return the highest } \text{MINIMAX-VALUE of } \text{SUCCESSORS}(state) \\
\text{else} \\
\quad \text{return the lowest } \text{MINIMAX-VALUE of } \text{SUCCESSORS}(state)
\]

**Complete:** ?  
**Optimal:** ?  
**Time complexity:** ?  
**Space complexity:** ?
Minimax: Recursive implementation

```python
function MINIMAX-DECISION(game) returns an operator
    for each op in OPERATORS[game] do
        Value[op] ← MINIMAX-VALUE(APPLY(op, game), game)
    end
    return the op with the highest Value[op]

function MINIMAX-VALUE(state, game) returns a utility value
    if TERMINAL-TEST(game)(state) then
        return UTILITY(game)(state)
    else if MAX is to move in state then
        return the highest MINIMAX-VALUE of Successors(state)
    else
        return the lowest MINIMAX-VALUE of Successors(state)
```

**Complete:** Yes, for finite state-space  
**Optimal:** Yes  
**Time complexity:** $O(b^m)$  
**Space complexity:** $O(bm)$ ( = DFS  
Does not keep all nodes in memory.)
Do We Have To Do All That Work?

MAX

MIN

3  12  8
Do We Have To Do All That Work?

MAX

MIN

3 12 8
Do We Have To Do All That Work?

Since 2 is smaller than 3, then there is no need for further search.
Do We Have To Do All That Work?

More on this next time: $\alpha$-$\beta$ pruning
1. Move evaluation without complete search

• Complete search is too complex and impractical

• **Evaluation function:** evaluates value of state using heuristics and cuts off search

• **New MINI MAX:**
  • **CUTOFF-TEST:** cutoff test to replace the termination condition (e.g., deadline, depth-limit, etc.)
  • **EVAL:** evaluation function to replace utility function (e.g., number of chess pieces taken)
Evaluation functions

- Weighted linear evaluation function: to combine \( n \) heuristics
  \[
  f = w_1 f_1 + w_2 f_2 + \ldots + w_n f_n
  \]

  E.g., \( w \)'s could be the values of pieces (1 for prawn, 3 for bishop etc.)
  \( f \)'s could be the number of type of pieces on the board
Note: exact values do not matter

Behaviour is preserved under any monotonic transformation of Eval

Only the order matters:

- payoff in deterministic games acts as an ordinal utility function
Minimax with cutoff: viable algorithm?

\texttt{MinimaxCutoff} is identical to \texttt{MinimaxValue} except
1. \texttt{Terminal?} is replaced by \texttt{Cutoff?}
2. \texttt{Utility} is replaced by \texttt{Eval}

Does it work in practice?

\[ b^m = 10^6, \quad b = 35 \quad \Rightarrow \quad m = 4 \]

4-ply lookahead is a hopeless chess player!

4-ply \approx \text{human novice}

8-ply \approx \text{typical PC, human master}

12-ply \approx \text{Deep Blue, Kasparov}

Assume we have 100 seconds, evaluate \(10^4\) nodes/s; can evaluate \(10^6\) nodes/move
2. $\alpha$-$\beta$ pruning: search cutoff

- **Pruning**: eliminating a branch of the search tree from consideration without exhaustive examination of each node

- **$\alpha$-$\beta$ pruning**: the basic idea is to prune portions of the search tree that cannot improve the utility value of the max or min node, by just considering the values of nodes seen so far.

- Does it work? Yes, in roughly cuts the branching factor from $b$ to $\sqrt{b}$ resulting in double as far look-ahead than pure minimax
α-β pruning: example

MAX

MIN

≥ 6

6

6 12 8
**α-β pruning: example**

MAX

MIN

\[ \begin{array}{c}
6 \\
12 \\
8 \\
\end{array} \]

\[ \begin{array}{c}
\geq 6 \\
\leq 2 \\
\times \\
\times \\
\end{array} \]
$\alpha$-$\beta$ pruning: example

\[ \text{MAX} \]

\[ \text{MIN} \]

\[ \geq 6 \]

\[ \leq 2 \]

\[ \leq 5 \]
α-β pruning: example

MAX

MIN

Selected move
**α-β pruning: general principle**

If $\alpha > v$ then MAX will choose $m$ so prune tree under $n$

*Similar for $\beta$ for MIN*
Properties of $\alpha$-$\beta$

Pruning *does not* affect final result

Good move ordering improves effectiveness of pruning

With “perfect ordering,” time complexity $= O(b^{m/2})$
  $\Rightarrow$ doubles depth of search
  $\Rightarrow$ can easily reach depth 8 and play good chess

A simple example of the value of reasoning about which computations are relevant (a form of *metareasoning*)
The $\alpha$-$\beta$ algorithm:

Basically $\text{MINIMAX} + \text{keep track of } \alpha, \beta + \text{prune}$

function $\text{Max-Value}(state, game, \alpha, \beta)$ returns the minimax value of $state$

inputs: $state$, current state in game

$game$, game description

$\alpha$, the best score for MAX along the path to $state$

$\beta$, the best score for MIN along the path to $state$

if $\text{Cutoff-Test}(state)$ then return $\text{Eval}(state)$

for each $s$ in $\text{Successors}(state)$ do

$\alpha \leftarrow \text{Max}(\alpha, \text{Min-Value}(s, game, \alpha, \beta))$

if $\alpha \geq \beta$ then return $\beta$

end

return $\alpha$

function $\text{Min-Value}(state, game, \alpha, \beta)$ returns the minimax value of $state$

if $\text{Cutoff-Test}(state)$ then return $\text{Eval}(state)$

for each $s$ in $\text{Successors}(state)$ do

$\beta \leftarrow \text{Min}(\beta, \text{Max-Value}(s, game, \alpha, \beta))$

if $\beta \leq \alpha$ then return $\alpha$

end

return $\beta$
More on the $\alpha$-$\beta$ algorithm

• Same basic idea as minimax, but prune (cut away) branches of the tree that we know will not contain the solution.
More on the $\alpha$-$\beta$ algorithm: start from Minimax

Basically Minimax + keep track of $\alpha$, $\beta$ + prune

```python
function Max-Value(state, game, $\alpha$, $\beta$) returns the minimax value of state
    inputs: state, current state in game
            game, game description
            $\alpha$, the best score for MAX along the path to state
            $\beta$, the best score for MIN along the path to state

    if Cutoff-Test(state) then return Eval(state)
    for each s in Successors(state) do
        $\alpha \leftarrow \operatorname{Max}(\alpha, \operatorname{Min-Value}(s, game, \alpha, \beta))$
        if $\alpha \geq \beta$ then return $\beta$
    end
    return $\alpha$

function Min-Value(state, game, $\alpha$, $\beta$) returns the minimax value of state

    if Cutoff-Test(state) then return Eval(state)
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        if $\beta \leq \alpha$ then return $\alpha$
    end
    return $\beta$
```
Remember: Minimax: Recursive implementation

function MINIMAX-DECISION(game) returns an operator

    for each op in OPERATORS[game] do
        Value[op] ← MINIMAX-VALUE(APPLY(op, game), game)
    end
	nreturn the op with the highest VALUE[op]

function MINIMAX-VALUE(state, game) returns a utility value

    if TERMINAL-TEST(game)(state) then
        return UTILITY(game)(state)
    else if MAX is to move in state then
        return the highest MINIMAX-VALUE of Successors(state)
    else
        return the lowest MINIMAX-VALUE of Successors(state)

Complete: Yes, for finite state-space  Time complexity: \(O(b^m)\)
Optimal: Yes  Space complexity: \(O(bm)\) (= DFS
Does not keep all nodes in memory.)
More on the \( \alpha-\beta \) algorithm

• Same basic idea as minimax, but prune (cut away) branches of the tree that we know will not contain the solution.

• Because minimax is depth-first, let’s consider nodes along a given path in the tree. Then, as we go along this path, we keep track of:
  • \( \alpha \): Best choice so far for MAX
  • \( \beta \): Best choice so far for MIN
More on the $\alpha$-$\beta$ algorithm: start from Minimax

Basically MINIMAX + keep track of $\alpha$, $\beta$ + prune

```plaintext
function Max-Value(state, game, $\alpha$, $\beta$) returns the minimax value of state
    inputs: state, current state in game
            game, game description
            $\alpha$, the best score for MAX along the path to state
            $\beta$, the best score for MIN along the path to state
    if Cutoff-Test(state) then return Eval(state)
    for each $s$ in Successors(state) do
        $\alpha \leftarrow \max(\alpha, \text{Min-Value}(s, game, \alpha, \beta))$
        if $\alpha \geq \beta$ then return $\beta$
    end
    return $\alpha$

function Min-Value(state, game, $\alpha$, $\beta$) returns the minimax value of state
    if Cutoff-Test(state) then return Eval(state)
    for each $s$ in Successors(state) do
        $\beta \leftarrow \min(\beta, \text{Max-Value}(s, game, \alpha, \beta))$
        if $\beta \leq \alpha$ then return $\alpha$
    end
    return $\beta$
```

Note: These are both Local variables. At the Start of the algorithm, We initialize them to $\alpha = -\infty$ and $\beta = +\infty$
More on the $\alpha$-$\beta$ algorithm

MAX

MIN

Max-Value loops over these

Min-Value loops over these

In Min-Value:

\[
\text{for each } s \text{ in } \text{Successors}(\text{state}) \text{ do}
\]
\[
\beta \leftarrow \text{MIN}(\beta, \text{MAX-VALUE}(s, \text{game}, \alpha, \beta))
\]
\[
\text{if } \beta \leq \alpha \text{ then return } \alpha
\]
\[
\text{return } \beta
\]
More on the $\alpha$-$\beta$ algorithm

In Max-Value:

```plaintext
for each s in SUCCESSORS(state) do
    $\alpha \leftarrow \text{MAX}(\alpha, \text{MIN-VALUE}(s, game, \alpha, \beta))$
    if $\alpha \geq \beta$ then return $\beta$
end
return $\alpha$
```

MAX

\[ \alpha = -\infty \]
\[ \beta = +\infty \]

MIN

Max-Value loops over these

\[ \alpha = 5 \]
\[ \beta = +\infty \]

```
5
\[ \alpha = -\infty \]
\[ \beta = 5 \]

10
\[ \alpha = -\infty \]
\[ \beta = 5 \]

6
\[ \alpha = -\infty \]
\[ \beta = 5 \]

2
\[ \alpha = -\infty \]

8
\[ \alpha = -\infty \]

7
\[ \alpha = -\infty \]
```

...
More on the $\alpha$-$\beta$ algorithm

In Min-Value:

```
for each s in Successors(state) do
    $\beta \leftarrow \text{MIN}(\beta, \text{MAX-VALUE}(s, game, \alpha, \beta))$
    if $\beta \leq \alpha$ then return $\alpha$
end
return $\beta$
```

MAX

MIN

Min-Value loops over these

MAX

5
$\alpha = -\infty$
$\beta = 5$

10
$\alpha = -\infty$
$\beta = 5$

6
$\alpha = -\infty$
$\beta = 5$

2
$\alpha = 5$
$\beta = 2$

8

7

End loop and return 5
More on the $\alpha$-$\beta$ algorithm

In Max-Value:

```
for each $s$ in SUCCESSORS(state) do
    $\alpha \leftarrow \text{MAX}(\alpha, \text{MIN-VALUE}(s, \text{game}, \alpha, \beta))$
    if $\alpha \geq \beta$ then return $\beta$
end
return $\alpha$
```

Max-Value loops over these

MAX

MIN

Max-Value loops over these

MAX

5 10 6

2 8 7

$\alpha = -\infty$ $\alpha = -\infty$ $\alpha = -\infty$

$\beta = 5$ $\beta = 5$ $\beta = 5$

$\alpha = 5$ $\alpha = 5$ $\alpha = 5$

$\beta = +\infty$ $\beta = +\infty$ $\beta = 2$

End loop and return 5
Another way to understand the algorithm

• From:

• For a given node N,

  $\alpha$ is the value of N to MAX
  $\beta$ is the value of N to MIN
MiniMax + Alpha–Beta

Example

Are Max Nodes
Are Min Nodes
$\alpha$-$\beta$ algorithm:

Basically $\text{MINIMAX} + \text{keep track of } \alpha, \beta + \text{prune}$

```plaintext
function Max-Value(state, game, $\alpha$, $\beta$) returns the minimax value of state
    inputs: state, current state in game
             game, game description
             $\alpha$, the best score for MAX along the path to state
             $\beta$, the best score for MIN along the path to state
    if Cutoff-Test(state) then return Eval(state)
    for each s in Successors(state) do
        $\alpha \leftarrow \text{Max}(\alpha, \text{Min-Value}(s, game, \alpha, \beta))$
        if $\alpha \geq \beta$ then return $\beta$
    end
    return $\alpha$

function Min-Value(state, game, $\alpha$, $\beta$) returns the minimax value of state
    if Cutoff-Test(state) then return Eval(state)
    for each s in Successors(state) do
        $\beta \leftarrow \text{Min}(\beta, \text{Max-Value}(s, game, \alpha, \beta))$
        if $\beta \leq \alpha$ then return $\alpha$
    end
    return $\beta$
```
### Solution

<table>
<thead>
<tr>
<th>NODE</th>
<th>TYPE</th>
<th>ALPHA</th>
<th>BETA</th>
<th>SCORE</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Max</td>
<td>-I</td>
<td>+I</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>Min</td>
<td>-I</td>
<td>+I</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>Max</td>
<td>-I</td>
<td>+I</td>
<td></td>
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<tr>
<td>D</td>
<td>Min</td>
<td>-I</td>
<td>+I</td>
<td></td>
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<tr>
<td>E</td>
<td>Max</td>
<td>10</td>
<td>10</td>
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<tr>
<td>F</td>
<td>Max</td>
<td>11</td>
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<td>11</td>
</tr>
<tr>
<td>G</td>
<td>Max</td>
<td>10</td>
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<td>10</td>
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<tr>
<td>H</td>
<td>Max</td>
<td>9</td>
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<tr>
<td>I</td>
<td>Max</td>
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<tr>
<td>M</td>
<td>Min</td>
<td>-I</td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>

...
State-of-the-art for deterministic games

Checkers: Chinook ended 40-year-reign of human world champion Marion Tinsley in 1994. Used an endgame database defining perfect play for all positions involving 8 or fewer pieces on the board, a total of 443,748,401,247 positions.


Othello: human champions refuse to compete against computers, who are too good.

Go: human champions refuse to compete against computers, who are too bad. In go, $b > 300$, so most programs use pattern knowledge bases to suggest plausible moves.
Nondeterministic games

E.g., in backgammon, the dice rolls determine the legal moves.

Simplified example with coin-flipping instead of dice-rolling:

\[
\begin{array}{c}
\text{MAX} \\
\text{CHANCE} \\
\text{MIN}
\end{array}
\]

\[
\begin{array}{c}
\text{MAX} \\
\text{CHANCE} \\
\text{MIN}
\end{array}
\]

\[
\begin{array}{c}
2 \\
4 \\
7
\end{array}
\]

\[
\begin{array}{c}
2 \\
4 \\
4
\end{array}
\]

\[
\begin{array}{c}
0 \\
-2
\end{array}
\]

\[
\begin{array}{c}
6 \\
0 \\
5
\end{array}
\]

\[
\begin{array}{c}
-2
\end{array}
\]

\[
\begin{array}{c}
0.5 \\
0.5 \\
0.5 \\
0.5
\end{array}
\]

\[
\begin{array}{c}
3 \\
-1
\end{array}
\]

CS 561, Sessions 8-9
Algorithm for nondeterministic games

**EXPECTIMINIMAX** gives perfect play

Just like **MINIMAX**, except we must also handle chance nodes:

...  
**if** `state` **is a chance node then**  
**return** average of **EXPECTIMINIMAX-VALUE** of **SUCCESSORS**(state)  
...

A version of $\alpha-\beta$ pruning is possible  
but only if the leaf values are bounded. **Why??**
Remember: Minimax algorithm

function MINIMAX-DECISION(game) returns an operator
    for each op in OPERATORS[game] do
        Value[op] ← MINIMAX-VALUE(APPLY(op, game), game)
    end
    return the op with the highest Value[op]

function MINIMAX-VALUE(state, game) returns a utility value
    if TERMINAL-TEST(game)(state) then
        return UTILITY(game)(state)
    else if MAX is to move in state then
        return the highest MINIMAX-VALUE of SUCCESSORS(state)
    else
        return the lowest MINIMAX-VALUE of SUCCESSORS(state)
Nondeterministic games: the element of chance

`expectimax` and `expectimin`, expected values over all possible outcomes
Nondeterministic games: the element of chance

Expectimax

MAX

Expectimin

MIN

\[ 4 = 0.5 \times 3 + 0.5 \times 5 \]
Evaluation functions: Exact values DO matter

Order-preserving transformation do not necessarily behave the same!
State-of-the-art for nondeterministic games

Dice rolls increase $b$: 21 possible rolls with 2 dice
Backgammon $\approx 20$ legal moves (can be 6,000 with 1-1 roll)

$$\text{depth } 4 = 20 \times (21 \times 20)^3 \approx 1.2 \times 10^9$$

As depth increases, probability of reaching a given node shrinks
$\Rightarrow$ value of lookahead is diminished

$\alpha-\beta$ pruning is much less effective
Summary

Games are fun to work on! (and dangerous)

They illustrate several important points about AI

◊ perfection is unattainable ⇒ must approximate

◊ good idea to think about what to think about

◊ uncertainty constrains the assignment of values to states

Games are to AI as grand prix racing is to automobile design
Exercise: Game Playing

Consider the following game tree in which the evaluation function values are shown below each leaf node. Assume that the root node corresponds to the maximizing player. Assume the search always visits children left-to-right.

(a) Compute the backed-up values computed by the minimax algorithm. Show your answer by writing values at the appropriate nodes in the above tree.

(b) Compute the backed-up values computed by the alpha-beta algorithm. What nodes will not be examined by the alpha-beta pruning algorithm?

(c) What move should Max choose once the values have been backed-up all the way?